# Announcements

1) # 4 on HW!

2) Colloquium Wednesday,
CB 2046, 3-4,
Mahesh Agarwal, on
elliptic curves and ellipses

3) matnorm file on
Canvas for HW #2

# Recall Full and Reduced SVD

$A \in \mathbb{C}^{m \times n}$

$$\boxed{A = \hat{U}\hat{\Sigma}(\hat{V})^*}$$

with $\hat{V}$ unitary $n \times n$, $\hat{\Sigma}$ diagonal $n \times n$, $\hat{U}$ orthonormal columns, $m \times n$ (reduced svd)

Full: $A = U \Sigma V^*$ with

$U$ unitary $m \times m$, $V = \hat{V}$,

and $\Sigma = \hat{\Sigma}$ followed

by $(m-n)$ rows of

zeros $(m \times n)$.

**Definition:** (singular vectors)

The right singular vectors of $A$ are the columns of $V$. The left singular vectors are the columns of $\hat{U}$.

# Geometric Description

Write $A = \hat{U}\hat{\Sigma}(\hat{V})^*$.

Applying $A$ to a right singular vector you get:

First, a standard basis element in $\mathbb{C}^n$ ($V$ unitary).

Second, a standard basis element scaled by $\hat{\Sigma}$.

Finally, a column of $U$, scaled by the appropriate entry in $\hat{\Sigma}$.

So if $A$ has full rank, it sends the "ellipsoid" in $\mathbb{C}^n$ determined by $\{v_1, v_2, \ldots, v_n\}$ to the ellipsoid in $\mathbb{C}^m$ determined by $(\sigma_1 u_1, \sigma_2 u_2, \ldots, \sigma_n u_n)$.

## Example 1:

$$A = \begin{bmatrix} -1 & 6 \\ 3 & 14 \end{bmatrix}.$$

Reduced svd (Matlab)

$$\hat{U} = \begin{bmatrix} .3743 & .9273 \\ .9273 & -.3743 \end{bmatrix}$$

$$\hat{\Sigma} = \begin{bmatrix} 15.4173 & 0 \\ 0 & 2.0756 \end{bmatrix}$$

$$\hat{V} = \begin{bmatrix} .1562 & -.9877 \\ .9877 & .1562 \end{bmatrix}$$

Recall: (rank) The rank of a matrix is the dimension of its column space = number of linearly independent rows = number of linearly independent columns.

**Theorem:** (unitaries and 2-norm)

If $A \in \mathbb{C}^{m \times n}$ and $V \in \mathbb{C}^{n \times n}$ and $U \in \mathbb{C}^{m \times m}$ are unitaries, then

$$\|A\|_2 = \|UAV\|_2 .$$

**Proof:** Unitaries preserve 2-norm. $\square$

**Theorem:** (SVD) Let $A \in \mathbb{C}^{m \times n}$.

1) The rank of $A$ is the number of nonzero singular values of $A$.

2) If $A = U \Sigma V^*$ (full svd), then the range of $A$ is the span of the first rank$(A)$ columns of $U$.

The kernel of $A$ is the span of the last $(n - \text{rank}(A))$ columns of $V$.

3) The 2-norm of $A$ is the largest singular value of $A$; the Frobenius norm is the 2-norm of the vector whose coordinates are the singular values of $A$.

4)* The singular values of $A$ are the square roots of the eigenvalues of $A^*A$.

5) If $n=m$, $|\det(A)|$ is equal to the product of the singular values of $A$.

proof: 3)

$$\|A\|_2 = \|U\Sigma V^*\|_2$$

$$= \|\Sigma\|_2$$ by previous theorem

= largest element on the diagonal

$$= \sigma_1$$

Similarly, you can show

$$\|\Sigma\|_F = \|U\Sigma V^*\|_F$$

$$= \|A\|_F .$$

$\square$

**Theorem:** (SVD)

1) rank

2) range and kernel

3) norms

4) Singular values and eigenvalues of $A^*A$

5) If $n=m$, $|\det(A)|$

**Theorem:** (rank-ones)

$$A = \sum_{j=1}^{r} \sigma_j u_j v_j^*$$

with $r = \text{rank}(A)$, and if

we set $A_\nu = \sum_{j=1}^{\nu} \sigma_j u_j v_j^*$,

and $0 \leq \nu \leq r$, then

1) $\|A - A_\nu\|_2$

$$= \min_{\substack{B \in \mathbb{C}^{m \times n} \\ \text{rank}(B) \le \nu}} \|A - B\|_2 = \sigma_{\nu+1}$$

with the convention that

$\sigma_{\nu+1} = 0$ if $\nu = n$.

2) $\|A - A_\nu\|_F = \min_{B \in \mathbb{C}^{m \times n}}$

$$= \min_{\substack{B \in \mathbb{C}^{m \times n} \\ \text{rank}(B) \le \nu}} \|A - B\|_F$$

$$= \sqrt{\sum_{j=\nu+1}^{n} \sigma_j^2}$$